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A Generalization of an Inequality of V. Markov to Multivariate Polynomials, II

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If P_m^r is a polynomial of total degree m ($\geqslant 2$) in $r (\geqslant 1)$ variables, then each of its coefficients with degree $m-1$ is bounded in absolute value by $\|P_m^r\|$ times a product of the absolute values of coefficients of univariate Chebyshev polynomials (the uniform norm is taken on the r-dimensional unit cube). This result generalizes a well-known inequality for univariate polynomials which is due to V. Markov. By a counterexample we demonstrate that such a bound does not hold for the coefficients with degree $\leqslant m-2$.

1. INTRODUCTION

A classical result of V. Markov [2] concerning the size of polynomial coefficients is the following set of sharp inequalities: If $P_m(x) = \sum_{k=0}^m a_k x^k$ is an arbitrary real-valued (univariate) polynomial with norm $\|P_m\| := \max |P_m(x)| \leqslant 1$, where $x \in I := [-1, 1]$, and $T_m(x) = \sum_{k=0}^m t_k^{(m)} x^k$ denotes the mth Chebyshev polynomial of the first kind with respect to I, then

$$|a_k| \leqslant \begin{cases} |t_k^{(m)}|, & \text{if } k \equiv m \bmod 2 \\ |t_k^{(m-1)}|, & \text{if } k \equiv m-1 \bmod 2 \end{cases} \tag{1}$$

(see also [3, p. 56] or [9, p. 167]). The integer coefficients $t_k^{(m)}$ are explicitly known (cf. [8, p. 32]). The case $k = m$ is originally due to Chebyshev [1]; see also [8, p. 57]:

$$|a_m| \leqslant 2^{m-1}. \tag{2}$$

Here we consider extensions of (1) to multivariate polynomials P_m^r of total degree $\leqslant m \in \mathbb{N}$ on the unit cube I^r, $r \geqslant 1$. The following notation will be used:

$$P_m^r(\mathbf{x}) = \sum_{|\mathbf{k}| \leqslant m} b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}, \qquad b_{\mathbf{k}} \in \mathbb{R}, \tag{3}$$

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with $\mathbf{x}=(x_1,\ldots,x_r)\in\mathbb{R}^r$, $\mathbf{k}=(k_1,\ldots,k_r)\in\mathbb{N}_0^r$, $\mathbf{x}^{\mathbf{k}}=x_1^{k_1}\cdots x_r^{k_r}$, and $|\mathbf{k}|=k_1+\cdots+k_r$. We put $\|P_m^r\|:=\max|P_m^r(\mathbf{x})|$, where $\mathbf{x}\in I^r$, and denote by $\mathbb{P}_m^r$ the linear space of polynomials given by (3). According to [7, Corollary 3] the following generalization of (2) holds:

$$|b_{\mathbf{k}}|\leqslant 2^{m-\bar{r}} \qquad \text{if } |\mathbf{k}|=m \text{ and } \|P_m^r\|\leqslant 1 \tag{4}$$

with equality if $P_m^r(\mathbf{x})=\prod_{q=1}^r T_{k_q}(x_q)$, where $\bar{r}$ denotes the number of nonvanishing components of $\mathbf{k}$. (An alternative proof of an extension of (4) is given in [4, Satz 1.5].)

Here we show that an estimate analogous to that in (4) holds for each $b_{\mathbf{k}}$ with $|\mathbf{k}|=m-1$ $(m\geqslant 2)$. By a counterexample we then demonstrate that neither products nor any rational functions of coefficients of (univariate) Chebyshev polynomials are enough to majorize the $b_{\mathbf{k}}$'s if $|\mathbf{k}|\leqslant m-2$.

2. Results and Proofs

We begin with an auxiliary result.

Lemma. Let $\mathbf{k}\in\mathbb{N}_0^r$ with $|\mathbf{k}|=m-1\in\mathbb{N}$ be arbitrary but fixed; let $\bar{\mathbb{P}}_m^r:=\operatorname{span}\{\mathbf{x}^{\mathbf{k}'}:|\mathbf{k}'|\leqslant m,\ \mathbf{k}'\neq\mathbf{k}\}$ denote that subspace of $\mathbb{P}_m^r$ whose basis does not contain the monomial $\mathbf{x}^{\mathbf{k}}$. Define sets $J_q:=\{0,\ldots,k_q+1\}\setminus\{k_q\}$ and

$$V:=\left\{\sum_{q=1}^r\sum_{j_q\in J_q} x_q^{j_q}G_{j_q}(x_1,\ldots,x_{q-1},x_{q+1},\ldots,x_r)\right\}, \tag{5}$$

where the G_{j_q}'s are continuous functions on I^{r-1}. Then the inclusion $\bar{\mathbb{P}}_m^r\subset V$ holds.

Proof. Because of the arbitrariness of the functions G_{j_q} it suffices to show that $\mathbf{x}^{\mathbf{k}'}\in V$ if $\mathbf{k}'\in\mathbb{N}_0^r$ and $|\mathbf{k}'|\leqslant m$ (but $\mathbf{k}'\neq\mathbf{k}$). The following observations concerning the components k_q' of $\mathbf{k}'$ and k_q of $\mathbf{k}$ are easy to verify: (i) if $|\mathbf{k}'|\leqslant m-1$ (but $\mathbf{k}'\neq\mathbf{k}$), then there exist k_q' and k_q with $k_q'<k_q$; (ii) if $|\mathbf{k}'|=m$, then there exist k_q' and k_q with $k_q'<k_q$ or with $k_q'=k_q+1$.

In those cases where $k_q'<k_q$ for some $q\in\{1,\ldots,r\}$ the monomial $\mathbf{x}^{\mathbf{k}'}$ can be written as $x_q^{k_q'}G(x_1,\ldots,x_{q-1},x_{q+1},\ldots,x_r)$ with a suitable continuous function G and $k_q'\in\{0,\ldots,k_q-1\}\subset J_q$. If $k_q'=k_q+1$ for some $q\in\{1,\ldots,r\}$ we may write $\mathbf{x}^{\mathbf{k}'}$ as $x_q^{k_q'}H(x_1,\ldots,x_{q-1},x_{q+1},\ldots,x_r)$ with a suitable continuous function H and $k_q'\in\{k_q+1\}\subset J_q$. In both cases $\mathbf{x}^{\mathbf{k}'}$ can be identified with an element of V. ∎

Theorem. *Let* $P_m^r\in\mathbb{P}_m^r$ *with* $\|P_m^r\|\leqslant 1$; *let* $\mathbf{k}\in\mathbb{N}_0^r$ *with* $|\mathbf{k}|=m-1\in\mathbb{N}$

be arbitrary but fixed and denote by $\bar{r}$ the number of nonvanishing components k_q of $\mathbf{k}$. *Then the coefficients $b_{\mathbf{k}}$ of P_m^r satisfy the estimate*

$$|b_{\mathbf{k}}| \leqslant 2^{m-\bar{r}-1} \qquad (|\mathbf{k}| = m-1) \tag{6}$$

with equality if $P_m^r(\mathbf{x}) = \prod_{q=1}^r T_{k_q}(x_q) \in \mathbb{P}_{m-1}^r \subset \mathbb{P}_m^r$.

Proof. Let $\bar{T}_m$ denote the mth Chebyshev polynomial normalized so that its leading coefficient is 1. A theorem of Markov (cf. [2, pp. 231, 237] or [3, p. 53]) states that

$$x_q^{k_q} - \bar{T}_{k_q}(x_q)$$

is the best L^∞-approximation to the monomial $x_q^{k_q}$ on I from the space span $\{1, x_q,..., x_q^{k_q-1}, x_q^{k_q+1}\}$. With the aid of Theorem 2.6.7 in [10] we infer from this that

$$\mathbf{x}^{\mathbf{k}} - \prod_{q=1}^{r} \bar{T}_{k_q}(x_q) \tag{7}$$

is a best L^∞-approximation to $\mathbf{x}^{\mathbf{k}}$ on I^r from the set V as defined in (5) and hence also from $\bar{\mathbb{P}}_m^r \subset V$ (see the preceding Lemma) since (7) belongs to $\bar{\mathbb{P}}_m^r$. The required estimate is then obtained as follows:

$$\begin{aligned} |b_{\mathbf{k}}| &\leqslant \|P_m^r\| \left(\inf_{\bar{P}_m^r \in \bar{\mathbb{P}}_m^r} \max_{\mathbf{x} \in I^r} |\mathbf{x}^{\mathbf{k}} - \bar{P}_m^r(\mathbf{x})| \right)^{-1} \\ &\leqslant \left\| \prod_{q=1}^{r} \bar{T}_{k_q} \right\|^{-1} = \left(\prod_{q=1}^{r} \|\bar{T}_{k_q}\| \right)^{-1} = 2^{m-\bar{r}-1} \end{aligned} \tag{8}$$

(cf. [9, Satz 1.2] or [11, p. 86]). ∎

In the light of the inequalities (1), (4) and (6) it is reasonable to ask whether the coefficients $b_{\mathbf{k}}$ in P_m^r with $\|P_m^r\| \leqslant 1$ will also be maximized by a product of coefficients of (univariate) Chebyshev polynomials if $|\mathbf{k}| \leqslant m-2$. If this were true one would have a complete multivariate analogue to Markov's inequalities (1). However, the answer is in the negative as we show by a counterexample.

EXAMPLE. Let $m=4$, $r=2$ and put $\mathbf{k} = (k_1, k_2) := (k, l)$ and $\mathbf{x} = (x_1, x_2) := (x, y)$. Our aim is to determine the largest coefficient $b_{(1,1)}$ (in absolute value) among all $P_4^2(x, y) = \sum_{0 \leqslant k+l \leqslant 4} b_{(k,l)} x^k y^l$ with $\|P_4^2\| \leqslant 1$. Observe that for $\mathbf{k} = (k, l) = (1, 1)$ we now have $|\mathbf{k}| = 2 = m-2$. It is interesting to note that the proof of our Theorem cannot be imitated here since $\bar{\bar{\mathbb{P}}}_4^2 := \operatorname{span}\{x^k y^l : 0 \leqslant k+l \leqslant 4, \ (k, l) \neq (1, 1)\}$ is no subset of the set V as defined in (5). In fact, we now have

$V=\{G_0(y)+x^2G_2(y)+H_0(x)+y^2H_2(x): G_0, G_2, H_0, H_2$ continuous on $I\}$ but from this set the monomials x^3y and xy^3 cannot be recovered. To reach our aim we shall apply the same reasoning as in (8). To this end, we have to compute a best L^∞-approximation to $f(x,y):=xy$ on I^2 from $\bar{\bar{\mathbb{P}}}{}^2_4$. This function is symmetric and odd in each of its variables; a best approximation to f with the same properties belongs necessarily to the one-dimensional subspace

$$W_A := \{w_A : w_A(x,y) = A(x^3y+xy^3), A \in \mathbb{R}\} \tag{9}$$

of $\bar{\bar{\mathbb{P}}}{}^2_4$. It suffices to determine a best approximation to f from W_A. Investigating the function

$$F_A := f - w_A \tag{10}$$

on I^2, partial differentiation yields the result that for

$$A = A' = (1+2^{1/2})/4, \tag{11}$$

$F_{A'}$ alternates at the eight points $(x,y)=(\pm 1, \pm 1)$ and $(x,y)=(\pm z, \pm z)$, where $z := (2^{1/2}-1)^{1/2}$. To be specific, $F_{A'}(x,y) = \|F_{A'}\| = (2^{1/2}-1)/2$ if

$$(x,y) \in M^+ := \{(z,z), (1,-1), (-z,-z), (-1,1)\} \subset I^2$$

and $F_{A'}(x,y) = -\|F_{A'}\|$ if

$$(x,y) \in M^- := \{(1,1), (z,-z), (-1,-1), (-z,z)\} \subset I^2.$$

There is no $w_A \in W_A$ which is positive on M^+ and negative on M^-. Indeed, if $w_A > 0$ on M^+ then in particular $w_A(z,z) > 0$ and hence $A > 0$. On the other hand, if $w_A < 0$ on M^- then in particular $w_A(1,1) < 0$ and hence $A < 0$, a contradiction.

We conclude from [10, Lemma 2.2.1] that $F_{A'}$ is the error-function of a best L^∞-approximation to f on I^2 from W_A. Hence we obtain

$$|b_{(1,1)}| \leqslant \|P_4^2\| \, \|F_{A'}\|^{-1} \leqslant \|F_{A'}\|^{-1} = 2(1+2^{1/2}), \tag{12}$$

with equality if $P_4^2(x,y) = 2(1+2^{1/2})\,F_{A'}(x,y)$. ∎

3. Remarks

(i) An alternative generalization of the cases $k=m$ and $k=m-1$ in Markov's inequalities (1) to the polynomial space $\mathbb{P}^r_m$ is to be found in [5].

(ii) A complete extension of (1) to multivariate polynomials is possible if we consider tensorproduct polynomials rather than polynomials with bounded total degree (cf. [6]).

(iii) The results presented here are excerpted from [4].

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